Math 2J
Lecture 12 - 10/24/12

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Principle Component Analysis (PCA)

- A method for decomposing complex data involving many attributes.
- Reveals the primary source of variation.
- Reveals any important relationships between attributes.

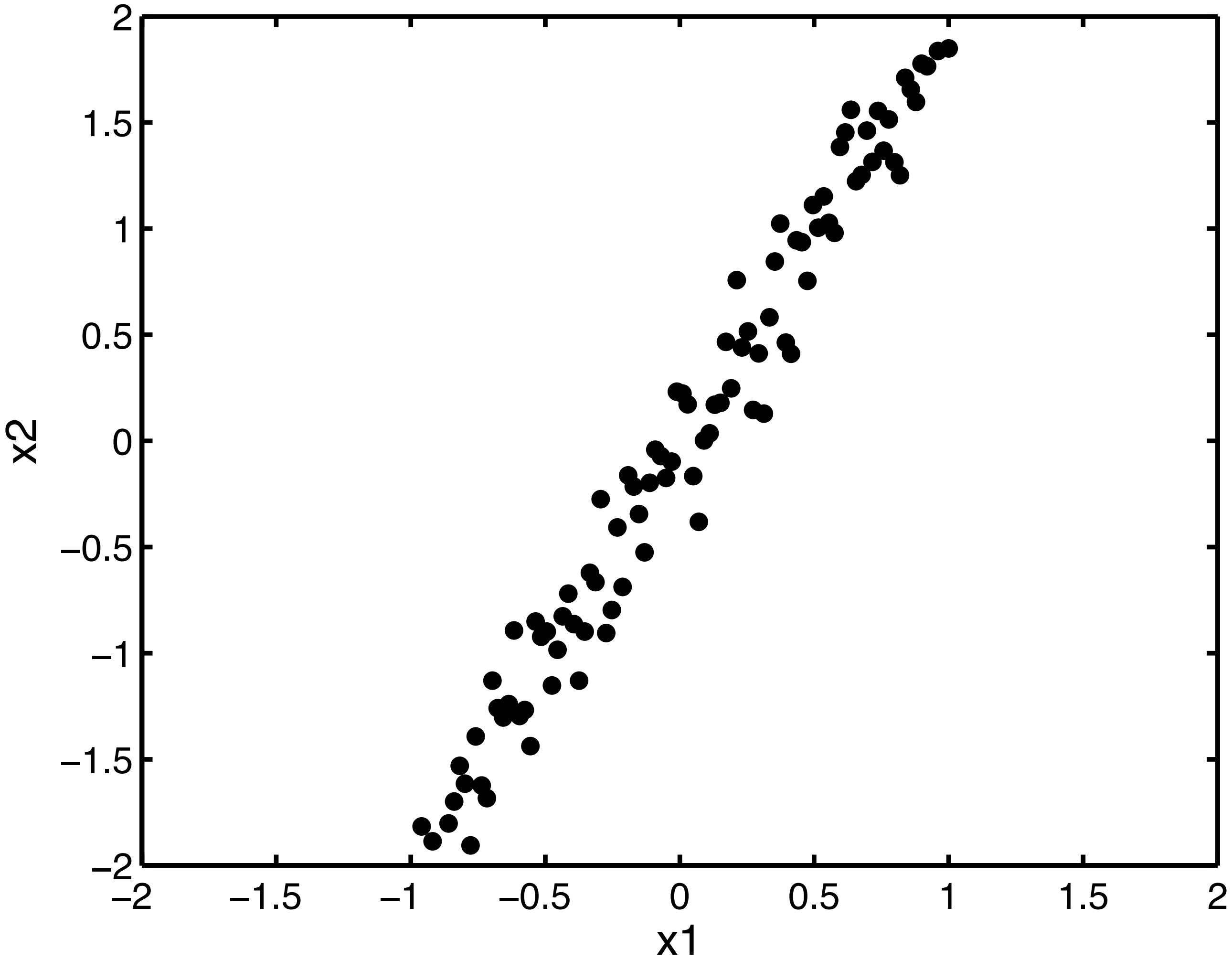
PCA Use

- Heavily used in:
  - Statistics
  - Image reconstruction, compression (think a .jpeg image)
  - Structural engineering

PCA Implementation

- Uses an eigenvalue / eigenvector decomposition to determine what aspects of data are most important.

Example



- This data clearly follows a trend line.
- PCA finds that trend line.

Step 1

- Compute the mean / average of each variable
- Superscripts index different observations.

$$\bar{x_1} = \frac{x_1^1 + x_1^2 + \ldots + x_1^n}{n} = \frac{\sum_1^n x_1^i}{n}$$

$$\bar{x_2} = \frac{x_2^1 + x_2^2 + \ldots + x_2^n}{n} = \frac{\sum_1^n x_2^i}{n}$$

Step 2

- Compute variance and co-variance

$$var(x_1) = \frac{(x_1^1 - \bar{x}_1)^2 + (x_1^2 - \bar{x}_1)^2 + \ldots + (x_1^n - \bar{x}_1)^2}{n-1}$$

$$var(x_2) = \frac{(x_2^1 - \bar{x}_2)^2 + (x_2^2 - \bar{x}_2)^2 + \ldots + (x_2^n - \bar{x}_2)^2}{n-1}$$

Step 2

- Compute variance and co-variance

$$var(x_1) = \frac{(x_1^1 - \bar{x}_1)^2 + (x_1^2 - \bar{x}_1)^2 + \ldots + (x_1^n - \bar{x}_1)^2}{n-1}$$

$$var(x_2) = \frac{(x_2^1 - \bar{x}_2)^2 + (x_2^2 - \bar{x}_2)^2 + \ldots + (x_2^n - \bar{x}_2)^2}{n-1}$$

$$cov(x_1, x_2) = \frac{(x_1^1 - \bar{x}_1)(x_2^1 - \bar{x}_2) + (x_1^2 - \bar{x}_1)(x_2^2 - \bar{x}_2) + \ldots + (x_1^n - \bar{x}_1)(x_2^n - \bar{x}_2)}{n-1}$$

$$cov(x_2, x_1) = \frac{(x_2^1 - \bar{x}_2)(x_1^1 - \bar{x}_1) + (x_2^2 - \bar{x}_2)(x_1^2 - \bar{x}_1) + \ldots + (x_2^n - \bar{x}_2)(x_1^n - \bar{x}_1)}{n-1}$$

Step 2

- These expressions describe how the observations differ from their average.

Step 3

- For a ***covariance matrix***

$$C = \begin{bmatrix} var(x_1) & cov(x_1, x_2) \\ cov(x_2, x_1) & var(x_2) \end{bmatrix}$$

- Notice that C is symmetric since cov(x1,x2)=cov(x2,x1).

Step 4

- In this case, the covariance matrix is

$$C = \begin{bmatrix} .34 & .69 \\ .69 & 1.42 \end{bmatrix}$$

- Now find the eigenvalues and eigenvectors.

$$C - \lambda\mathbb{I} = \begin{bmatrix} .34 - \lambda & .69 \\ .69 & 1.42 - \lambda \end{bmatrix}$$

$$\lambda^2 - 1.76\lambda + .0067 = 0$$

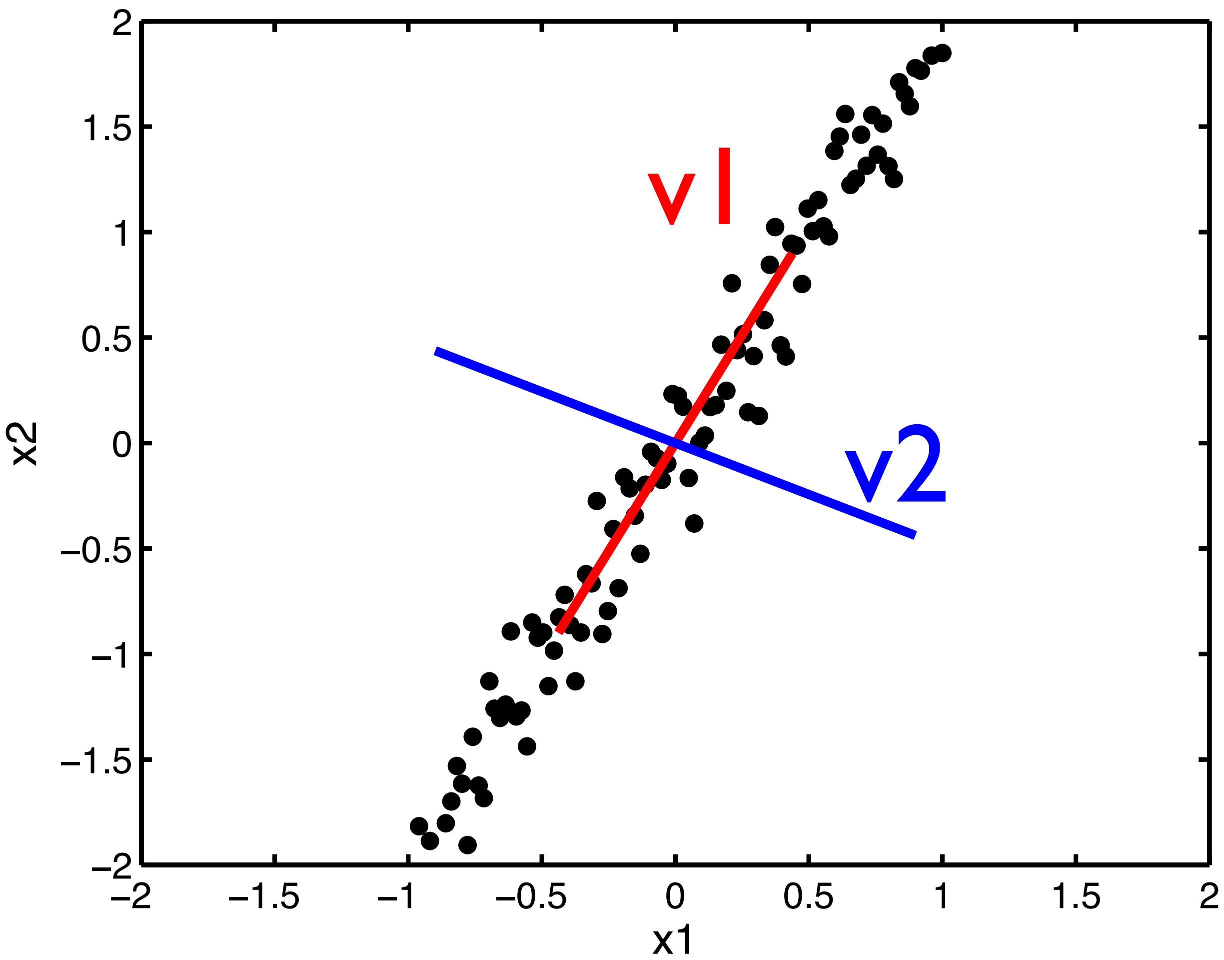
Step 4

$$\lambda_1 = .0074 \qquad \lambda_2 = 1.7588$$

$$\vec{v}_1 = [-.89, \quad .44] \qquad \vec{v}_2 = [.44, \quad .89]$$

- Large eigenvalues indicate the eigenvector is important.
- Small eigenvalues indicate little of the data variation occurs in that direction.

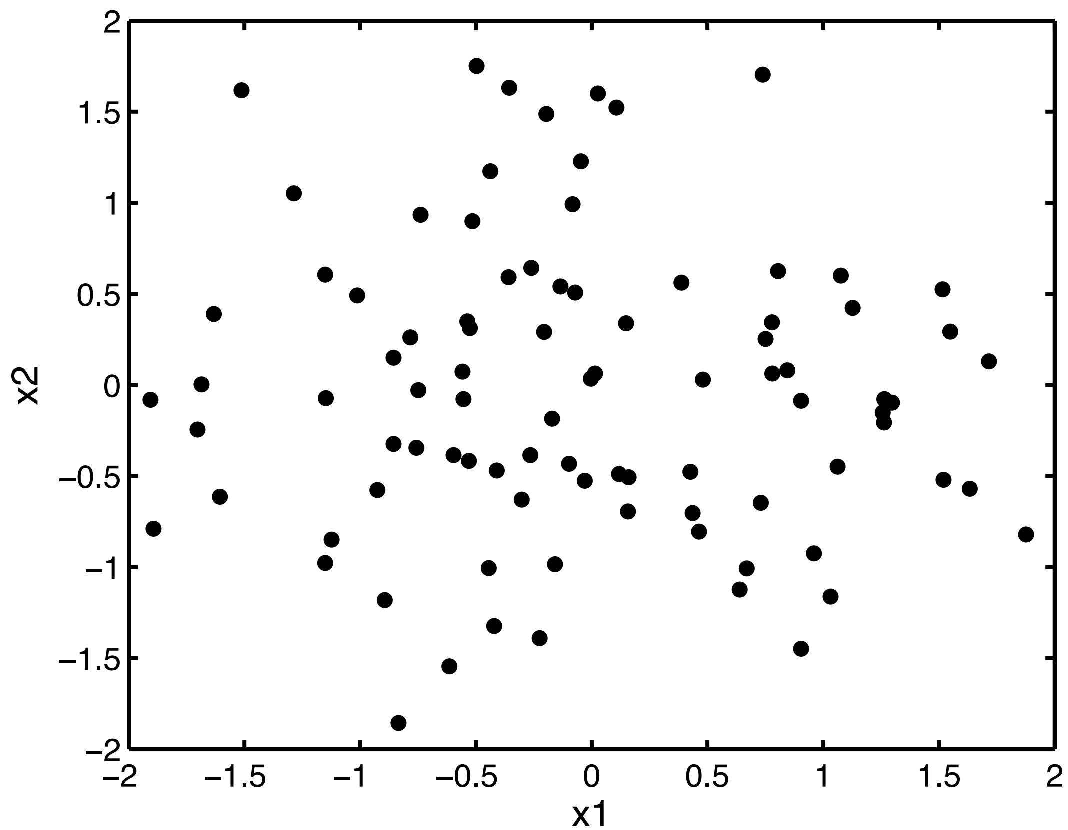
End Result



- So the "dominant" eigenvector captures the data trend.
- Can Say $x_2=2*x_1$

$$\vec{v}_2 = [.44, \quad .89]$$

Another example

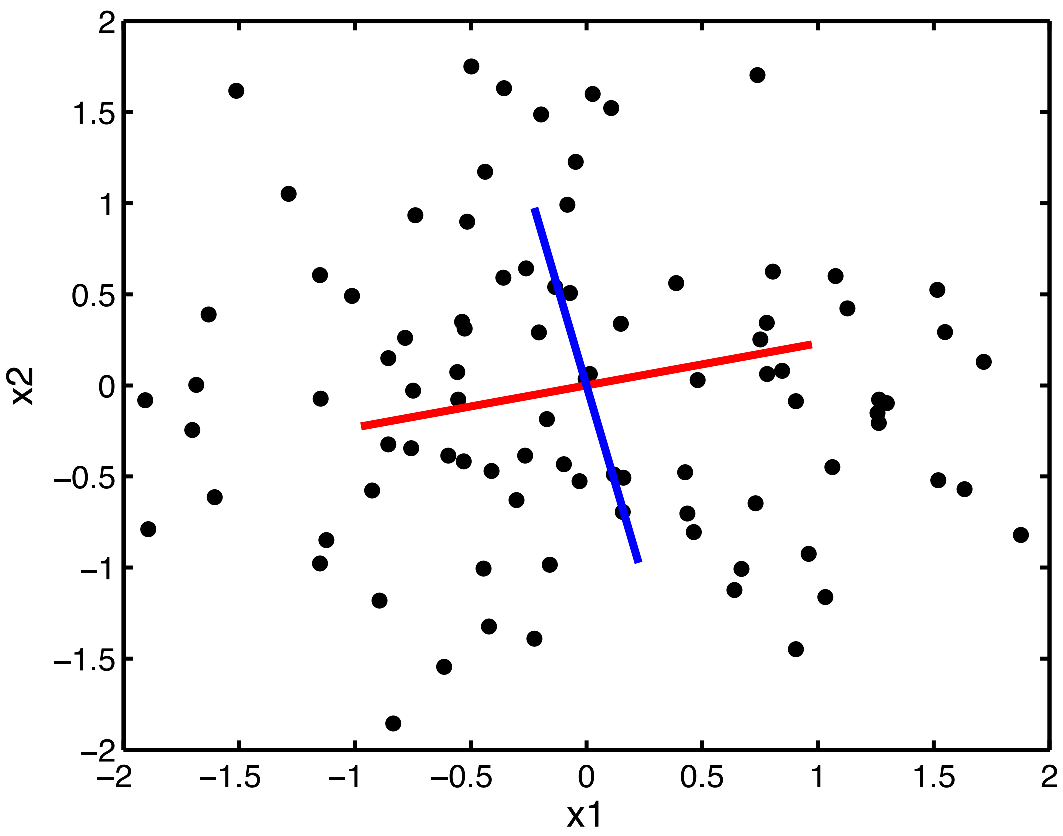


- Data appears much more random and less corollated.
- PCA will tell you this.

PCA Results

$$\lambda_1 = .91$$
$$\lambda_2 = 1.05$$

- Nether eigenvalue is dominant.
- So there is no "trend" in the data



PCA Recap

- Large eigenvalues are associated with “directions” that account for a lot of data variation.
  - This suggests a correlation between variables.
- Small eigenvalues are associated with directions that account for very little data variation.